Maxwell's Equations

A sing-a-long science song written by Lynda Williams. Dedicated to Dr. Susan Lea who helped me through graduate E&M (Jackson.) Maxwell's equations are 4 mathematical equations that relate the Electric Field (E) and magnetic field (B) to the charge (ρ) and current (J) densities that specify the fields and give rise to electromagnetic radiation - light. In the song we use Gaussian units. Lyrics in parenthesis are the phonetic reading of the equations.

Today's Lesson: The Genesis of light.

In the beginning there was Maxwell's Equations, two flux and two curl, obeying charge conservation.

And then there was light.

Lesson One: Maxwell's Equations with sources in free space

Equation One: Gauss'Law for the Electric Field

The flux of the E Field through a closed surface
(The integral of E dot ds)
Is due to the charge density contained inside
(4 pi integral of rho dV)

$$\text{Flux} = \Phi_E = \oint_S \vec{E} \bullet d\bar{S}$$

$$4\pi \int_V \rho dV$$

Put it all together, it reads:
(Surface integral of E is equal to
4 pi volume integral of rho dV)

$$\oint_S \vec{E} \bullet d\bar{S} = 4\pi \int_V \rho dV$$

Recall the divergence theorem for a vector A
(Closed surface integral of A is equal to the volume integral of the divergence of A)

$$\oint_S \bar{A} \bullet d\bar{S} = \int_V \bar{\nabla} \bullet \bar{A} dV$$

and apply it to Gauss' Law for E
(The surface integral of E is equal to the volume integral of the divergence of E which is equal to 4 pi volume integral of rho dV)

$$\oint_S \bar{E} \bullet d\bar{S} = \int_V \bar{\nabla} \bullet \bar{E} dV = 4\pi \int_V \rho dV$$

Since the integrals are equal for any volume the integrands are equal too, giving us the differential form of the Law:
(del dot E is 4 pi rho)
Say it! (del dot E is 4 pi rho)
repetez! (del dot E is 4 pi rho)
one more time! (del dot E is 4 pi rho)

$$\int_V \bar{\nabla} \bullet \bar{E} dV = 4\pi \int_V \rho dV$$

$$\bar{\nabla} \bullet \bar{E} = 4\pi\rho$$

What does it mean?
The flux of the E field though a closed surface is due to the charge density contained inside!
Electric charges produce electric fields!

Maxwell's Equations! Our salvation!

Equation Two: Gauss' Law for the Magnetic Field

The B field is a dipole field so no matter how small the volume is you will always find equal numbers of north and south poles. So if you integrate over a closed surface you'll always get a net magnetic flux of zero. In integral form it is:
(Closed surface integral of B dot dS is zero)

$$Flux = \Phi_B = \oint_S \overline{B} \bullet d\overline{S} = 0$$

Use the Divergence theorem,
(Surface integral of A is equal to the volume integral of the divergence of A)
and apply it to Gauss' Law for B

$$\oint_S \overline{A} \bullet d\overline{S} = \int_V \overline{\nabla} \bullet \overline{A} dV$$

(Closed surface integral of B is equal to the volume integral of the divergence of B which is equal to zero)

$$\oint_S \overline{B} \bullet d\overline{S} = \int_V \overline{\nabla} \bullet \overline{B} dV = 0$$

Since the integrals are equal for any volume the integrands are equal too, giving us the differential form of the Law:
(del dot B is equal to zero)
say it!
(del dot B is equal to zero)
repetez
(del dot B is equal to zero)
one more time
(del dot B is equal to zero)

$$\int_V \overline{\nabla} \bullet \overline{B} dV = 0$$

$$\overline{\nabla} \bullet \overline{B} = 0$$

What does it mean?
The flux of the B field through
a closed surface is zero and
no matter how much we wish
magnetic monopoles do not exist!

Maxwell's Equations!
Our salvation!

Equation Three: Faraday's Law

Since del dot B is exactly zero, we have an interesting result. If we don't close the surface integral we get a magnetic flux. And a magnetic flux that changes in time produces an emf, that is, a non conservative circulating E field with a nonzero closed line integral.

$$Flux = \Phi_B = \int_S \overline{B} \bullet d\overline{S} \neq 0$$

$$\varepsilon mf = -\frac{1}{c}\frac{d\Phi_B}{dt} = -\frac{1}{c}\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S}$$

$$\varepsilon mf = \oint_C \overline{E} \bullet d\overline{l} \neq 0$$

It is: (Line integral of E is minus one over c d dt integral of B)

$$\oint_C \overline{E} \bullet d\overline{l} = -\frac{1}{c}\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S}$$

Recall Stoke's Theorem for a vector A
(Closed line integral of A is equal to the open surface integral of the curl of A)

$$\oint_C \overline{A} \bullet d\overline{l} = \int_S \overline{\nabla} \times \overline{A} \bullet d\overline{S}$$

Apply it to the integral form of the Law :
(closed line integral of E is equal to the open surface integral of curl of E which minus one over c d dt integral of B)

$$\oint_C \overline{E} \bullet d\overline{l} = \int_S \overline{\nabla} \times \overline{E} \bullet d\overline{S}$$
$$= -\frac{1}{c}\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S}$$

Since the integrals are equal for any surface, the integrands are equal too, giving us the differential form of the Law:

$$\int_S \overline{\nabla} \times \overline{E} \bullet d\overline{S} = -\frac{1}{c}\frac{d}{dt}\int_S \overline{B} \bullet d\overline{S}$$

$$\overline{\nabla} \times \overline{E} = -\frac{1}{c}\frac{\partial \overline{B}}{\partial t}$$

(del cross E is minus one over c partial B partial t)
say it!
(del cross E is minus one over c partial B partial t)
repetez!
(del cross E is minus one over c partial B partial t)
one more time!
(del cross E is minus one over c partial B partial t)

What does it mean?
A magnetic field that is changing in time produces a non-conservative E field!

Maxwell's Equations!
Our Salvation!

Equation Four: Ampere's Law with Conservation

The line integral of the B field around a closed path is equal to the surface integral of the current density flow through a surface bound by the path.
In integral form:
(Closed line integral of B is equal to 4 pi over c surface integral of J)

$$\oint_C \overline{B} \bullet d\overline{l} = \frac{4\pi}{c}\int_S \overline{J} \bullet d\overline{S}$$

Once again we use Stoke's Theorem:
(Line integral of A is equal to the surface integral of the curl of A)

$$\oint_C \overline{A} \bullet d\overline{l} = \int_S \overline{\nabla} \times \overline{A} \bullet d\overline{S}$$

And apply it to the integral form of the Law:
(line integral of B is equal to the open surface integral of curl of B which is equal to 4 pi over c surface integral of J)

$$\oint_C \overline{B} \bullet d\overline{l} = \int_S \overline{\nabla} \times \overline{B} \bullet d\overline{S}$$
$$= \frac{4\pi}{c}\int_S \overline{J} \bullet d\overline{S}$$

Since the integrals are equal for any surface, the integrands are equal too, giving us the differential form of the Law:

$$\int_S \nabla \times \overline{B} \bullet d\overline{S} = \frac{4\pi}{c} \int_S \overline{J} \bullet d\overline{S}$$

(del cross B is equal to 4 pi over c J)
say it!
(del cross B is equal to 4 pi over c J)
repetez!
(del cross B is equal to 4 pi over c J)
one more time!
(del cross B is equal to 4 pi over c J)

$$\overline{\nabla} \times \overline{B} = \frac{4\pi}{c} \overline{J}$$

But that's not all! If you take the Divergence of Ampere's Law - well do it!
(del dot del cross B equals del dot 4 pi over c J)
We have a problem because the divergence of a curl is zero but the the divergence of J is not!

$$\overline{\nabla} \bullet \overline{\nabla} \times \overline{B} = \overline{\nabla} \bullet \frac{4\pi}{c} \overline{J}$$

$$\overline{\nabla} \bullet \overline{\nabla} \times \overline{B} = 0 \neq \overline{\nabla} \bullet \frac{4\pi}{c} \overline{J} = -\frac{4\pi}{c} \frac{\partial \rho}{\partial t}$$

Recall the equation of continuity, that is:
(del dot J is equal to minus partial rho partial t)

$$\overline{\nabla} \bullet \overline{J} = -\frac{\partial \rho}{\partial t}$$

That is net outflow of current is equal to the rate at wich the charges are lost. That's charge conservation! We must obey it!

But we know that a changing E field produces a B field and if you take the partial time derivative of Gauss' electric law you get a current term.

$$\frac{\partial}{\partial t} \overline{\nabla} \bullet \overline{E} = \frac{\partial}{\partial t} 4\pi \rho$$

$$\overline{\nabla} \bullet \frac{\partial \overline{E}}{\partial t} = 4\pi \frac{\partial \rho}{\partial t}$$

So let us define a 'Displacement Current'
(partial E partial t)

$$\frac{\partial \overline{E}}{\partial t}$$

and put it into Ampere's equation
so that it obeys charge conservation

(del cross B is equal to 4 pi over c J plus one over c partial E partial t)
say it!
(del cross B is equal to 4 pi over c J plus one over c partial E partial t)
repetez!
(del cross B is equal to 4 pi over c J plus one over partial E partial t)
one more time!
(del cross B is equal to 4 pi over c J plus one over c partial E partial t)

$$\overline{\nabla} \times \overline{B} = \frac{4\pi}{c} \overline{J} + \frac{1}{c} \frac{\partial \overline{E}}{\partial t}$$

What does it mean?
The curl of the B field is
due to the current flow and
a changing electric field.

Maxwell’s Equations!
Our Salvation!

Finally, we have it all.
say it with me
in the light of the law!

Gauss!
(del dot E is 4 pi rho)

$$\overline{\nabla} \bullet \overline{E} = 4\pi\rho$$

NO monopoles!
(del dot B is equal to zero)

$$\overline{\nabla} \bullet \overline{B} = 0$$

Faraday.
(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla} \times \overline{E} = -\frac{1}{c}\frac{\partial \overline{B}}{\partial t}$$

Ampere's
(del cross B is equal to 4 pi over c J plus
one over partial E partial t)

$$\overline{\nabla} \times \overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial \overline{E}}{\partial t}$$

Gauss!
(del dot E is equal to 4 pi rho)

$$\overline{\nabla} \bullet \overline{E} = 4\pi\rho$$

NO monopoles!
(del dot B is equal to zero)

$$\overline{\nabla} \bullet \overline{B} = 0$$

Faraday.
(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla} \times \overline{E} = -\frac{1}{c}\frac{\partial \overline{B}}{\partial t}$$

Ampere's Law
(del cross B is equal to 4 pi over c J plus
one over partial E partial t)

$$\overline{\nabla} \times \overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial \overline{E}}{\partial t}$$

In the beginning of the world
was Maxwell's Equations
Two flux and Two curl
obeying charge conservation
and then there was light….alright!

Lesson #2: Maxwell's Equations in Macroscopic

Media

The free space equations are not valid in the presence of matter, in macroscopic media because the E and B fields produce polarization (P) and magnetization (M) effects in the bound charges of the material.

P, the polarization vector, is the electric dipole density induced by the external field, E. P weakens E so we define the Displacement D which is the field due only to charges that are free .
(D is equal to E plus 4 pi P)
say it, D: (D is equal to E plus 4 pi P)

$$\vec{D} = \vec{E} + 4\pi \vec{P}$$

M, the magnetization, is the magnetic dipole density induced by the external field, B.
M strengthens B and so we define the H which is the field due only to the currents that are free.
(H is equal to B minus 4 pi M)
say it, H: (H is equal to B minus 4 pi M)

$$\vec{H} = \vec{B} - 4\pi \vec{M}$$

H and D come into play where you have a rho or J.
Just substitute D for E, do the same, H for B.

Gauss's Law becomes
(del dot D equals 4 pi rho)
say it. (del dot D equals 4 pi rho)

$$\vec{\nabla} \bullet \vec{D} = 4\pi\rho$$

Ampere's Law becomes
(del cross H is equal to 4 pi over c J plus 1 over c partial D partial t)

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

again
(del cross H is equal to 4 pi over c J plus 1 over c partial D partial t)

Finally we have it all.

Gauss: (del dot D is equal to 4 pi rho)

$$\vec{\nabla} \bullet \vec{D} = 4\pi\rho$$

NO Magnetic Monopoles:
(del dot B is equal to zero)

$$\vec{\nabla} \bullet \vec{B} = 0$$

Faraday: (del cross E is equal to minus one over c partial B partial t)

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Ampere's Law:(del cross H is equal to 4 pi over c J plus 1 over c partial D partial t)

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$$

Once again my friend.

Gauss: (del dot D is equal to 4 pi rho)

$$\overline{\nabla}\bullet\overline{D} = 4\pi\rho$$

NO Monopoles:
(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

Faraday: (del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

Ampere's Law with conservation: (del cross H is equal to 4 pi over c J plus 1 over c partial D partial t)

$$\overline{\nabla}\times\overline{H} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial\overline{D}}{\partial t}$$

In the beginning world
was Maxwell's Equations
Two flux and Two curl
obeying charge conservation
and then there was light….
alright!

Lesson #3: Maxwell's Equations in Vacuum

You thought it was over but now its time to begin. What happens when there are no currents, no charges within? Then everything simplifies in this special case we have Maxwell's equations in empty space!

Since there's no sources set the J's and rhos to zero. H's turn into to B's and D's turn back into E's!

$$\rho = 0$$
$$J = 0$$

Let's start with gauss:
(Del dot E is equal to zero)

$$\overline{\nabla}\bullet\overline{E} = 0$$

(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

(del cross E is is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

(del cross B is equal to one over c partial E partial t)

$$\overline{\nabla}\times\overline{B} = \frac{1}{c}\frac{\partial\overline{E}}{\partial t}$$

Once again!
(Del dot E is equal to zero)

$$\overline{\nabla}\bullet\overline{E} = 0$$

(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

(del cross B is equal to one over c partial E partial t)

Once again $\overline{\nabla}\times\overline{B}=\frac{1}{c}\frac{\partial\overline{E}}{\partial t}$

(del dot E is equal to zero) $\overline{\nabla}\bullet\overline{E}=0$

(del dot B is equal to zero) $\overline{\nabla}\bullet\overline{B}=0$

(del cross E is equal to minus one over c partial B partial t) $\overline{\nabla}\times\overline{E}=-\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$

(del cross B is equal to one over c partial E partial t) $\overline{\nabla}\times\overline{B}=\frac{1}{c}\frac{\partial\overline{E}}{\partial t}$

Zero flux, no monopoles,
change in B produces E
change in E produces B
Oh! The symmetry!

In the beginning of the world
was Maxwell's Equations
Two flux and Two curl
and then there was light….
alright!

Lesson # 4: The Test!
Now its time for the test.
Just relax and do your best!
Don't worry if at first you get it wrong.
This is just a song!
The answers will be given at the end
and you can do it again and again
until you get it right,
Maxwell's equations of Light! Alright!

Maxwell's Equations in free space with sources!
Gauss! ( )
No monopoles! ( )
Faraday! ( )
Ampere's Law with conservation! ( )

Maxwell's Equations in Macroscopic Media
Gauss! ( )
No monopoles! ( )
Faraday! ( )
Ampere's Law with conservation! ( )

Maxwell's Equations in Empty Space
Gauss! ( )
No monopoles! ( )
Faraday! ( )
Ampere's Law! ( )

Very good! But how did you do?
Try it again…my friend…

Free space with sources:
Gauss: (del dot E is equal to 4 pi rho)

$$\overline{\nabla}\bullet\overline{E} = 4\pi\rho$$

No Monopoles ( )
(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

Faraday: ( )
(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

Ampere's Law with conservation: ( )
(del cross B is equal to 4 pi over c J plus 1 over c partial E partial t)

$$\overline{\nabla}\times\overline{B} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial\overline{E}}{\partial t}$$

Macroscopic Media

Gauss: ( )
(del dot D is equal to 4 pi rho)

$$\overline{\nabla}\bullet\overline{D} = 4\pi\rho$$

No Monopoles:
(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

Faraday: ( )
(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

Ampere: ( )
(del cross H is equal to 4 pi over c J plus 1 over c partial D partial t)

$$\overline{\nabla}\times\overline{H} = \frac{4\pi}{c}\overline{J} + \frac{1}{c}\frac{\partial\overline{D}}{\partial t}$$

Maxwell's Equations in Empty Space
Gauss: ( )
(del dot D is equal zero)

$$\overline{\nabla}\bullet\overline{E} = 0$$

No Monopoles ( )
(del dot B is equal to zero)

$$\overline{\nabla}\bullet\overline{B} = 0$$

Faraday: ( )
(del cross E is equal to minus one over c partial B partial t)

$$\overline{\nabla}\times\overline{E} = -\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

Ampere: ( )
(del cross B is equal to 1 over c partial E partial t)

In the beginning of the world
was Maxwell's Equations
Two flux and Two curl
obeying charge conservation
and then there was light!
alright!

$$\overline{\nabla}\times\overline{B}=\frac{1}{c}\frac{\partial\overline{E}}{\partial t}$$

Epilogue: The Wave Equation

Start with Maxwell's Equation in a Vacuum

$$\overline{\nabla}\bullet\overline{E}=0$$
$$\overline{\nabla}\bullet\overline{B}=0$$
$$\overline{\nabla}\times\overline{E}=-\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$
$$\overline{\nabla}\times\overline{B}=\frac{1}{c}\frac{\partial\overline{E}}{\partial t}$$

and take the curl of Faraday's Law.

$$\overline{\nabla}\times\left(\overline{\nabla}\times\overline{E}\right)=-\overline{\nabla}\times\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$

The trick is to use the vector ID for the curl of a curl and then you'll see that everything simplifies cuz Gauss's Law tells us del dot E is zero

$$\overline{\nabla}\times\left(\overline{\nabla}\times\overline{A}\right)=\overline{\nabla}\left(\overline{\nabla}\cdot\overline{A}\right)-\nabla^2\overline{A}$$

$$\overline{\nabla}\times\left(\overline{\nabla}\times\overline{E}\right)=\overline{\nabla}\left(\overline{\nabla}\cdot\overline{E}\right)-\nabla^2\overline{E}$$
$$\overline{\nabla}\times\left(\overline{\nabla}\times\overline{E}\right)=-\nabla^2\overline{E}$$

and Ampere's law simplifies the other side

$$-\frac{\partial}{\partial t}\frac{1}{c}\overline{\nabla}\times\overline{B}$$
$$-\frac{\partial}{\partial t}\frac{1}{c}\overline{\nabla}\times\overline{B}=-\frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2}$$

Put it all together and we derive a wave equation for the E field:

$$\overline{\nabla}\times\left(\overline{\nabla}\times\overline{E}\right)=-\overline{\nabla}\times\frac{1}{c}\frac{\partial\overline{B}}{\partial t}$$
$$-\nabla^2\overline{E}=-\frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2}$$

our familiar friend!

$$\nabla^2\overline{E}-\frac{1}{c^2}\frac{\partial^2\overline{E}}{\partial t^2}=0$$

Take the curl of Amperes Law and follow the same path you'll get an equation for B too.

$$\nabla^2\overline{B}-\frac{1}{c^2}\frac{\partial^2\overline{B}}{\partial t^2}=0$$

These are both wave equations that describe transverse, plane waves traveling at the speed of c - the speed of light!

$$\nabla^2 \overline{E} - \frac{1}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} = 0$$

$$\nabla^2 \overline{B} - \frac{1}{c^2}\frac{\partial^2 \overline{B}}{\partial t^2} = 0$$

A changing E produces B
A changing B produces E
Electromagnetic Fields
Oscillating and regenerating
at the speed of light.
Electromagnetic waves
travelling at the speed of light.

In the beginning of the world
was maxwell's equations
two flux and two curl
obeying conservation
and then there was light!