Maxwell's Equations: *The Wave Equation*

A sing-a-long science song written and performed by Lynda Williams to the tune of Madonna's *Ray of Light*.
(Maxwell's equations are 4 mathematical equations that relate the Electric Field (E) and magnetic field (B) to the charge (ρ) and current (J) densities that specify the fields. In the song we use Gaussian units)

Epilogue: The Wave Equation

Start with Maxwell's Equation in a Vacuum

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

and take the curl of Faraday's Law.

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\vec{\nabla}\times\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

The trick is to use the vector ID
for the curl of a curl and then you'll see that
everything simplifies cuz Gauss's Law tells us
that del dot E is zero
and Ampere's law simplifies the other side

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \nabla^2\vec{E}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = -\nabla^2\vec{E}$$

$$-\nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

Put it all together and we derive
a wave equation for the E field:

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Our familiar friend!

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

Take the curl of Amperes Law and follow the
same path you'll get an equation for B too.

Now isn't this interesting!
These are both wave equations that describe
transverse, plane waves traveling at the speed
of c – the speed of light!
Transverse, plane waves traveling, traveling,
at the speed of light!

$$u(x,t) = e^{i\vec{k}\cdot\vec{x}-\omega t}$$

A changing E produces B
A changing B produces E
Electromagnetic Fields
Oscillating and regenerating
at the speed of light.

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

Electromagnetic waves
travelling at the speed of light.

In the beginning of the world
was maxwell's equations
two flux and two curl
obeying conservation
and then there was light!

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$